

Note on ultraviolet renormalization and ground state energy of the Nelson model

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July 21, 2015

Abstract

Ultraviolet (UV) renormalization of the Nelson model H_ε in quantum field theory is considered. E. Nelson proved that $\lim_{\varepsilon \rightarrow 0} e^{-T(H_\varepsilon - E_\varepsilon^{\text{ren}})}$ converges to $e^{-TH^{\text{ren}}}$ in [Nel64a]. A relationship between a ultraviolet renormalization term $E_\varepsilon^{\text{ren}}$ and the ground state energy $E_\varepsilon(g^2)$ of the Hamiltonian with total momentum zero $H_\varepsilon(0)$ is studied by functional integrations. Here g denotes a coupling constant involved in $H_\varepsilon(0)$. It can be derived from the formula

$$E_\varepsilon(g^2) = - \lim_{T \rightarrow \infty} \frac{1}{2T} \log(\mathbb{1}, e^{-2TH_\varepsilon(0)} \mathbb{1})$$

that $E_\varepsilon^{\text{ren}}$ coincides with the coefficient of g^2 in the expansion of $E_\varepsilon(g^2)$ in g^2 , i.e., $E_\varepsilon^{\text{ren}} = \lim_{g \rightarrow 0} E_\varepsilon(g^2)/g^2$, and $E_\varepsilon(g^2) - g^2 E_\varepsilon^{\text{ren}}$ converges as ultraviolet cutoff is removed.

1 The Nelson model

In this paper we consider a relationship between a ultraviolet (UV) renormalization and the ground state energy of the Nelson model in quantum field theory *by functional integrations*. The Nelson model describes an interaction system between a scalar bose field and particles governed by a Schrödinger operator with an external potential. We prepare tools used in this paper. The boson Fock space \mathcal{F} over $L^2(\mathbb{R}^3)$ is defined by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} [\otimes_s^n L^2(\mathbb{R}^3)]. \quad (1.1)$$

Here $\otimes_s^n L^2(\mathbb{R}^3)$ describes n fold symmetric tensor product of $L^2(\mathbb{R}^3)$ with $\otimes_s^0 L^2(\mathbb{R}^3) = \mathbb{C}$. Let $a^*(f)$ and $a(f)$, $f \in L^2(\mathbb{R}^3)$, be the creation operator and the annihilation operator, respectively, in \mathcal{F} , which satisfy $(a^*(f))^* = a(f)$ and canonical commutation

relations:

$$[a(f), a^*(g)] = (\bar{f}, g)\mathbb{1}, \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)]. \quad (1.2)$$

Note that (f, g) denotes the scalar product on $L^2(\mathbb{R}^3)$ and it is linear in g and anti-linear in f . We also note that $f \mapsto a^*(f)$ and $f \mapsto a(f)$ are linear. Denote the dispersion relation by $\omega(k) = |k|$. Then the free field Hamiltonian H_f of \mathcal{F} is then defined by the second quantization of ω , i.e., $H_f = d\Gamma(\omega) = \int \omega(k) a^*(k) a(k) dk$. It satisfies that

$$e^{-itH_f} a^*(f) e^{-itH_f} = a^*(e^{-it\omega} f), \quad e^{-itH_f} a(f) e^{-itH_f} = a(e^{it\omega} f). \quad (1.3)$$

Hence it follows that

$$[H_f, a(f)] = -a(\omega f), \quad [H_f, a^*(f)] = -a^*(\omega f).$$

Furthermore for the Fock vacuum $\mathbb{1}_{\mathcal{F}} = 1 \oplus 0 \oplus 0 \cdots \in \mathcal{F}$, it follows that $H_f \mathbb{1}_{\mathcal{F}} = 0$.

Definition 1.1 *The Nelson Hamiltonian H is a self-adjoint operator acting in the Hilbert space $L^2(\mathbb{R}^3) \otimes \mathcal{F} \cong L^2(\mathbb{R}^3, \mathcal{F})$, which is given by*

$$H = \left(-\frac{1}{2}\Delta + V\right) \otimes \mathbb{1} + \mathbb{1} \otimes H_f + g\phi, \quad (1.4)$$

where $g \in \mathbb{R}$ is a coupling constant, $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ an external potential, the interaction is defined by $(\phi\Phi)(x) = \phi(x)\Phi(x)$ for a.e. $x \in \mathbb{R}^3$ and the field operator $\phi(x)$ by

$$\phi(x) = \frac{1}{\sqrt{2}} \left(a^*(\hat{\varphi}/\sqrt{\omega} e^{i(\cdot, x)}) + a(\tilde{\varphi}/\sqrt{\omega} e^{-i(\cdot, x)}) \right) \quad (1.5)$$

with $\tilde{\varphi}(k) = \hat{\varphi}(-k)$. Let H_0 be the operator defined by H with coupling constant g replaced by 0. We have to mention the self-adjointness of H . Suppose that

$$\hat{\varphi}/\sqrt{\omega}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^3), \quad \hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}. \quad (1.6)$$

Then the interaction H_I is well defined, symmetric and infinitesimally H_0 -bounded, i.e., for arbitrary $\varepsilon > 0$, there exists a $b_\varepsilon > 0$ such that

$$\|H_I \Phi\| \leq \varepsilon \|H_0 \Phi\| + b_\varepsilon \|\Phi\|$$

for all $\Phi \in D(H_0)$. Thus H is self-adjoint on $D(H_0)$ by the Kato-Rellich theorem. Throughout this paper we assume condition (1.6).

2 UV renormalization and ground state energy

A point charge limit of H , $\hat{\varphi}(k) \rightarrow \mathbb{1}$, is studied in [Nel64a, Nel64b] and recently in [GHPS12, GHL14, Hir15]. Let $\lambda > 0$ be a strictly positive infrared cutoff parameter and we fix it throughout. This assumption is used in the proof of Lemma 3.7. Consider the cutoff function

$$\hat{\varphi}_\varepsilon(k) = e^{-\varepsilon|k|^2/2} \mathbb{1}_{|k| \geq \lambda}, \quad \varepsilon > 0, \quad (2.1)$$

and define the regularized Hamiltonian by

$$H_\varepsilon = \left(-\frac{1}{2}\Delta + V\right) \otimes \mathbb{1} + \mathbb{1} \otimes H_f + g\phi_\varepsilon, \quad \varepsilon > 0, \quad (2.2)$$

where ϕ_ε is defined by ϕ with $\hat{\varphi}$ replaced by $\hat{\varphi}_\varepsilon$. Here $\varepsilon > 0$ is regarded as the UV cutoff parameter. Let

$$E_\varepsilon^{\text{ren}} = -g^2 \int_{|k| > \lambda} \frac{e^{-\varepsilon|k|^2}}{2\omega(k)} \beta(k) dk, \quad (2.3)$$

where β is given by

$$\beta(k) = \frac{1}{\omega(k) + |k|^2/2}. \quad (2.4)$$

Notice that $E_\varepsilon^{\text{ren}} \rightarrow -\infty$ as $\varepsilon \downarrow 0$. E. Nelson proved the proposition below in [Nel64a].

Proposition 2.1 *There exists a constant C such that $H_{\text{ren}} - E_\varepsilon^{\text{ren}} > -C$ uniformly in ε and there exists a self-adjoint operator H_{ren} such that*

$$s - \lim_{\varepsilon \downarrow 0} e^{-T(H_\varepsilon - g^2 E_\varepsilon^{\text{ren}})} = e^{-TH_{\text{ren}}}. \quad (2.5)$$

PROOF. Refer to see [Nel64a]. □

Let $V = 0$. Then H_ε is translation invariant, i.e.,

$$[H_\varepsilon, P_{\text{tot}, \mu}] = 0, \quad \mu = 1, 2, 3,$$

where P_{tot} is the total momentum defined by $P_{\text{tot}} = -i\nabla \otimes \mathbb{1} + \mathbb{1} \otimes P_f$. Here P_f denotes the field momentum operator given by $P_f = d\Gamma(k) = \int k a^*(k) a(k) dk$. Thus H_ε can be decomposed as $H_\varepsilon = \int_{\mathbb{R}^3}^\oplus H_\varepsilon(P) dP$, where

$$H_\varepsilon(P) = \frac{1}{2}(P - P_f)^2 + H_f + g\phi_\varepsilon(0) \quad (2.6)$$

is a self-adjoint operator in \mathcal{F} for each $P \in \mathbb{R}^3$. Let $E_\varepsilon(g^2) = \inf \sigma(H_\varepsilon(0))$ be the bottom of the spectrum of the Nelson model with zero-total momentum, $P = 0 \in \mathbb{R}^3$.

Suppose that formally $E_\varepsilon(g^2)$ can be expanded in g^2 as $E_\varepsilon(g^2) = E_\varepsilon(0) + a_2g^2 + a_4g^4 + \dots$, and the ground state energy as $\varphi_g = \mathbb{1} + g\phi_1 + g^2\phi_2 + \dots$. Note that $E_\varepsilon(0) = 0$. Then from equation $H_\varepsilon(0)\varphi_g = E_\varepsilon(g^2)\varphi_g$, we can derive the identity $\phi_1 = -(\frac{1}{2}P_f^2 + H_f)^{-1}\phi_\varepsilon(0)\mathbb{1}_{\mathcal{F}}$ and

$$a_2 = -(\mathbb{1}_{\mathcal{F}}, \phi_\varepsilon(0)\phi_1) = -(\phi_\varepsilon(0)\mathbb{1}_{\mathcal{F}}, (\frac{1}{2}P_f^2 + H_f)^{-1}\phi_\varepsilon(0)\mathbb{1}_{\mathcal{F}}) = E_\varepsilon^{\text{ren}}.$$

Hence $a_2 = E_\varepsilon^{\text{ren}}$ is derived. Furthermore we expect that a_n , $n \geq 4$, converges as $\varepsilon \downarrow 0$, and hence

$$\lim_{\varepsilon \downarrow 0} |E_\varepsilon(g^2) - g^2 E_\varepsilon^{\text{ren}}| < \infty. \quad (2.7)$$

All the statements mentioned above are however *informal*. In this paper we are concerned with these facts by functional integrations in non-pertubative way. We can show (2.7) for arbitrary values of g , and $\lim_{g \rightarrow 0} E_\varepsilon(g^2)/g^2 = E_\varepsilon^{\text{ren}}$ in Theorem 3.4.

Remark 2.2 In Proposition 2.1, (2.7) is proven by an operator theory, we however prove this by applying functional integrations.

3 Functional integrations

Let $(B_t)_{t \in \mathbb{R}}$ denote the 3-dimensional Brownian motion on $C(\mathbb{R}, \mathbb{R}^3)$ with the Wiener measure W . $\mathbb{E}[\dots]$ denotes the expectation with respect to W describing the Wiener measure starting from $0 \in \mathbb{R}^3$.

Lemma 3.1 For $P \in \mathbb{R}^3$, it follows that

$$(\mathbb{1}_{\mathcal{F}}, e^{-2TH_\varepsilon(P)}\mathbb{1}_{\mathcal{F}}) = \mathbb{E} \left[e^{iP \cdot (B_T - B_{-T})} e^{\frac{g^2}{2} S_\varepsilon} \right], \quad (3.1)$$

where

$$S_\varepsilon = \int_{-T}^T ds \int_{-T}^T dt W_\varepsilon(B_t - B_s, t - s) \quad (3.2)$$

and $W_\varepsilon : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$W_\varepsilon(x, t) = \int_{|k| \geq \lambda} \frac{e^{-\varepsilon|k|^2} e^{-ik \cdot x} e^{-\omega(k)|t|}}{2\omega(k)} dk. \quad (3.3)$$

PROOF. Refer to see [Hir15, Lemma 2.2]. □

Putting $P = 0$ in Lemma 3.1, we have

$$(\mathbb{1}_{\mathcal{F}}, e^{-2TH_\varepsilon(0)}\mathbb{1}_{\mathcal{F}}) = \mathbb{E} \left[e^{\frac{g^2}{2} S_\varepsilon} \right]. \quad (3.4)$$

Lemma 3.2 *Let $\lambda > 0$. Then*

$$E_\varepsilon(g^2) = - \lim_{T \rightarrow \infty} \frac{1}{2T} \log(\mathbb{1}, e^{-2TH_\varepsilon(0)} \mathbb{1}). \quad (3.5)$$

In particular

$$E_\varepsilon(g^2) = - \lim_{T \rightarrow \infty} \frac{1}{2T} \log \mathbb{E} \left[e^{\frac{g^2}{2} S_\varepsilon} \right]. \quad (3.6)$$

PROOF. Since $\lambda > 0$, it is shown that $H_\varepsilon(0)$ has the unique ground state φ_g and it is strictly positive. See also Appendix. Hence $(\mathbb{1}, \varphi_g) > 0$. In particular $(\mathbb{1}, \varphi_g) \neq 0$. Thus (3.5) follows. \square

It can be seen that the pair potential $W_\varepsilon(B_t - B_s, t - s)$ is singular at the diagonal part $t = s$. We shall remove the diagonal part by using the Itô formula, which is done in [GHL14]. We introduce the function

$$\varrho_\varepsilon(x, t) = \int_{|k| \geq \lambda} \frac{e^{-\varepsilon|k|^2} e^{-ik \cdot x - \omega(k)|t|}}{2\omega(k)} \beta(k) dk, \quad \varepsilon \geq 0. \quad (3.7)$$

Lemma 3.3 *It follows that*

$$S_\varepsilon = S_\varepsilon^{\text{ren}} + 4T \varrho_\varepsilon(0, 0), \quad \varepsilon > 0,$$

where

$$\begin{aligned} S_\varepsilon^{\text{ren}} &= S_\varepsilon^{OD} + 2 \int_{-T}^T \left(\int_s^{[s+\tau]} \nabla \varrho_\varepsilon(B_t - B_s, t - s) \cdot dB_t \right) ds \\ &\quad - 2 \int_{-T}^T \varrho_\varepsilon(B_{[s+\tau]} - B_s, [s+\tau] - s) ds. \end{aligned} \quad (3.8)$$

Here $0 < \tau < T$ is an arbitrary number, and $[t] = -T \vee t \wedge T$, S_ε^{OD} denotes the off-diagonal part given by

$$S_\varepsilon^{OD} = 2 \int_{-T}^T ds \int_{[s+\tau]}^T W_\varepsilon(B_t - B_s, t - s) dt$$

and the integrand is

$$\nabla_\mu \varrho_\varepsilon(X, t) = \int_{|k| \geq \lambda} \frac{-ik_\mu e^{-ik \cdot X} e^{-|t|\omega(k)} e^{-\varepsilon|k|^2}}{2\omega(k)} \beta(k) dk.$$

PROOF. It is shown by the Itô formula that

$$\int_s^S W_\varepsilon(B_t - B_s, t - s) dt = \varrho_\varepsilon(0, 0) - \varrho_\varepsilon(B_S - B_s, S - s) + \int_s^S \nabla \varrho_\varepsilon(B_t - B_s, t - s) \cdot dB_t. \quad (3.9)$$

Then the lemma follows directly. \square

$-\varrho_\varepsilon(0, 0)$ can be regarded as the diagonal part of W_ε and turns to be a renormalization term. I.e., we have the lemma below.

Lemma 3.4 *Let $\varepsilon > 0$. Then $E_\varepsilon^{\text{ren}} = -\varrho_\varepsilon(0, 0)$.*

Lemma 3.5 *There exist constants $b > 0$ and $c > 0$ independent of g such that for all $\varepsilon > 0$,*

$$\mathbb{E} \left[e^{\frac{g^2}{2} S_\varepsilon^{\text{ren}}} \right] \leq e^{b(c+g^4 T + g^2 \log T) + c(\tau)(g^2/2)T}, \quad (3.10)$$

where

$$c(\tau) = 8\pi \int_\lambda^\infty e^{-\varepsilon r^2} e^{-\tau r} dr. \quad (3.11)$$

PROOF. Let $S_\varepsilon^{\text{ren}} = S_\varepsilon^{\text{OD}} + Y + Z$, where

$$\begin{aligned} Y &= 2 \int_{-T}^T \left(\int_s^{[s+\tau]} \nabla \varrho_\varepsilon(B_t - B_s, t - s) dB_t \right) ds, \\ Z &= -2 \int_{-T}^T \varrho_\varepsilon(B_{[s+\tau]} - B_s, [s+\tau] - s) ds. \end{aligned}$$

It is established in [GHL14, Lemma 2.10] that

$$\mathbb{E}[e^{\alpha Y}] \leq e^{\alpha^2 T b_1} \quad (3.12)$$

with some constant b_1 . We estimate $\mathbb{E}[e^{\alpha Z}]$. Straightforwardly there exists a constant $M > 0$ such that $|\varrho_\varepsilon(B_T - B_s, T - s)| \leq |\varrho_\varepsilon(0, T - s)| < M$ for all T , and

$$\varrho_\varepsilon(0, T - s) \leq \frac{1}{2} e^{-\lambda|T-s|}.$$

Then we have

$$\begin{aligned} |Z| &\leq 2 \int_0^{2T} du \varrho_\varepsilon(0, u) = 2 \left(\int_0^1 + \int_1^{2T} \right) du \varrho_\varepsilon(0, u) \\ &\leq 2M + \int_1^{2T} du \frac{1}{u} = 2M + \log(2T) - 1. \end{aligned} \quad (3.13)$$

Finally we can compute $S_\varepsilon^{\text{OD}}$. We have

$$\begin{aligned} |S_\varepsilon^{\text{OD}}| &\leq 2 \int_{-T}^{T-\tau} ds \int_{s+\tau}^T dt \int_{|k| \geq \lambda} \frac{1}{2\omega(k)} e^{-\varepsilon|k|^2} e^{-\omega(k)|t-s|} dk \\ &= 4\pi \int_\lambda^\infty e^{-\varepsilon r^2} \frac{e^{-\tau r}}{r} (e^{-(2T-\tau)r} - 1 + (2T-\tau)r) dr \leq c(\tau)T. \end{aligned} \quad (3.14)$$

Then bound (3.10) follows from (3.12), (3.13), (3.14) and the Schwarz inequality $\mathbb{E}[e^{(g^2/2)(S_\varepsilon^{\text{OD}}+Y+Z)}] \leq \mathbb{E}[e^{g^2 Y}]^{1/2} \mathbb{E}[e^{g^2(S_\varepsilon^{\text{OD}}+Z)}]^{1/2}$. \square

Remark 3.6 Constants b and c given in Lemma 3.5 also depend on τ . See [GHL14, Lemma 2.8, Lemma 2.10, (2.36)].

Now we state the key lemma.

Lemma 3.7 *Let $b > 0$ and $c(\tau)$ be those in Lemma 3.5. Then*

$$\left| \frac{E_\varepsilon(g^2)}{g^2} + \varrho_\varepsilon(0, 0) \right| \leq \frac{1}{2}(g^2 b + \frac{1}{2}c(\tau)). \quad (3.15)$$

PROOF. By Lemmas 3.1 and 3.2 we have

$$E_\varepsilon(g^2) = - \lim_{T \rightarrow \infty} \frac{1}{2T} \log \mathbb{E} \left[e^{\frac{g^2}{2}(S_\varepsilon^{\text{ren}} + 4T\varrho_\varepsilon(0,0))} \right]. \quad (3.16)$$

We then have

$$E_\varepsilon(g^2) = -g^2 \varrho_\varepsilon(0, 0) - \lim_{T \rightarrow \infty} \frac{1}{2T} \log \mathbb{E} \left[e^{\frac{g^2}{2} S_\varepsilon^{\text{ren}}} \right]$$

Hence

$$|E_\varepsilon(g^2) + g^2 \varrho_\varepsilon(0, 0)| \leq \lim_{T \rightarrow \infty} \frac{1}{2T} \log \mathbb{E} \left[e^{\frac{g^2}{2} S_\varepsilon^{\text{ren}}} \right].$$

By Lemma 3.5 we can obtain (3.15). □

We now state the main theorem in this paper.

Theorem 3.8 *It follows that*

$$\lim_{g \rightarrow 0} \frac{E_\varepsilon(g^2)}{g^2} = E_\varepsilon^{\text{ren}} \quad (3.17)$$

and

$$\lim_{\varepsilon \downarrow 0} |E_\varepsilon(g^2) - g^2 E_\varepsilon^{\text{ren}}| < \infty. \quad (3.18)$$

PROOF. By Lemmas 3.4 and 3.7 we see that

$$\left| \frac{E_\varepsilon(g^2)}{g^2} - E_\varepsilon^{\text{ren}} \right| \leq \frac{1}{2}(g^2 b + \frac{1}{2}c(\tau)). \quad (3.19)$$

Take $g \rightarrow 0$. We have

$$\lim_{g \rightarrow 0} \left| \frac{E_\varepsilon(g^2)}{g^2} - E_\varepsilon^{\text{ren}} \right| \leq \frac{1}{4}c(\tau) \quad (3.20)$$

holds for arbitrary $\tau > 0$. $\lim_{\tau \rightarrow \infty} c(\tau) = 0$ implies (3.17). Furthermore (3.18) can be derived from (3.19) and the fact $\lim_{\varepsilon \downarrow 0} c(\tau) < \infty$. □

A Existence of the ground state

For the self-consistency of the paper we show the uniqueness and the existence of ground state of $H_\varepsilon(0)$. The proof mentioned below is taken from [Hir15, Lemma 2.9]. Let $\varphi_g^T = e^{-TH_\varepsilon(0)} \mathbb{1}_{\mathcal{F}} / \|e^{-TH_\varepsilon(0)} \mathbb{1}_{\mathcal{F}}\|$ and $\gamma(T) = (\mathbb{1}_{\mathcal{F}}, \varphi_g^T)^2$, i.e.,

$$\gamma(T) = \frac{(\mathbb{1}_{\mathcal{F}}, e^{-TH_\varepsilon(0)} \mathbb{1}_{\mathcal{F}})^2}{(\mathbb{1}_{\mathcal{F}}, e^{-2TH_\varepsilon(0)} \mathbb{1}_{\mathcal{F}})}. \quad (\text{A.1})$$

Proposition A.1 *For all $\varepsilon > 0$ and $\lambda > 0$, $H_\varepsilon(0)$ has the ground state and it is unique.*

PROOF. The uniqueness follows from the fact that $e^{-tH_\varepsilon(0)}$ is positivity improving. It remains to show the existence of ground state. The useful criteria is as follows. There exists a ground state of $H_\varepsilon(0)$ if and only if $\lim_{T \rightarrow \infty} \gamma(T) > 0$ [LMS02]. Thus it is enough to show $\lim_{T \rightarrow \infty} \gamma(T) > 0$. By Lemma 3.1 we have

$$\gamma(T) = \frac{\left(\mathbb{E}[e^{\frac{g^2}{2} \int_0^T dt \int_0^T ds W_\varepsilon}] \right)^2}{\mathbb{E}[e^{\frac{g^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_\varepsilon}]}.$$

Here $W_\varepsilon = W_\varepsilon(B_t - B_s, t - s)$. By the reflection symmetry and the Markov property of the Brownian motion we have

$$\gamma(T) = \frac{\mathbb{E}[e^{\frac{g^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_\varepsilon - g^2 \int_{-T}^0 dt \int_0^T W_\varepsilon}]}{\mathbb{E}[e^{\frac{g^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_\varepsilon}]}.$$

By estimating $\int_{-T}^0 dt \int_0^T W_\varepsilon$ straightforwardly, we have

$$\gamma(T) \geq \exp \left(-g^2 \int_{\mathbb{R}^3} \mathbb{1}_{|k| \geq \lambda} \frac{e^{-\varepsilon|k|^2}}{\omega(k)^3} dk \right) > 0 \quad (\text{A.2})$$

for all $T > 0$. Note that $\lambda > 0$. Then the proposition follows. \square

Acknowledgments: The author acknowledges support of Challenging Exploratory Research 15K13445 from JSPS.

References

- [GHPS12] C. Gérard, F. Hiroshima, A. Panati and A. Suzuki, Removal of the UV cutoff for the Nelson model with variable coefficients, *Lett Math Phys*, **101** (2012), 305–322.

- [GHL14] M. Gubinelli, F. Hiroshima and J. Lorinczi, Ultraviolet renormalization of the Nelson Hamiltonian through functional integration, *J. Funct. Anal.* **267** (2014), 3125–3153.
- [Hir15] F. Hiroshima, Translation invariant models in QFT without ultraviolet cut-offs, arXiv:1506.07514, preprint 2015.
- [LMS02] J. Lórinzi, R.A.Minlos and H. Spohn, The infrared behavior in Nelson’s model of quantum particle coupled to a massless scalar field, *Ann. Henri Poincaré* **3** (2002), 1–28.
- [Nel64a] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964), 1190–1197.
- [Nel64b] E. Nelson, Schrödinger particles interacting with a quantized scalar field, in: *Proc. Conference on Analysis in Function Space*, W. T. Martin and I. Segal (eds.), p. 87, MIT Press, 1964.